GLOBAL COLLEGE OF MANAGEMENT

DEPARTMENT OF MATHEMATICS

# BASIC MATHEMATICS

CLASS-XI

# COMPLEX NUMBER

## Introduction:

suppose the equation $x^2 = 4$

then $x = \pm\sqrt{4} = \pm 2$. Real solutions are possible

Now

suppose the equation $x^2 = -4$

then $x = \pm\sqrt{-4}$

In this case , we can not find the square root of -ve numbers.

To find the solution, we define

$$i^2 = -1$$

Then $i$ is called imaginary unit. Then the solution is

$$x = \pm\sqrt{-4} = \pm\sqrt{(2i)^2} = \pm 2i$$

So those numbers which contains imaginary unit($i$) is called the complex number.

## Definition:

A complex number is an ordered of the form $(a, b)$.

where 'a' is called the real part of the complex number and

'b' is called the imaginary part of the complex number.

NOTE: $(a, b) = a + i.b$ [we prove it later]

i is called imaginary unit.

Complex numbers are denoted by z $or\ w$.

Example:

| Complex number | Real part | Imaginary part |
|---|---|---|
| (3,4) = 3 + 4i | 3 | 4 |
| (2, -5) = 2 - i.5 | 2 | -5 |
| (1,-1) = 1 – i | 1 | -1 |
| (0,7) = 7i = 0+7i | 0 | 7 |
| $(\sqrt{2}, \sqrt{3}) = \sqrt{2} + \sqrt{3}\, i$ | $\sqrt{2}$ | $\sqrt{3}$ |
| (0,1) = i =0+i | 0 | 1 |

## Imaginary unit:

A complex number of the form $(0,1)$ is called imaginary unit.

It is denoted by i.

**NOTE:**

In complex number $z = (a, b)$ if $b = 0$

Then $z = (a, 0)$ is a purely real number and we write

$(a, 0) = a$

**Examples:**

$3 = (3,0)$

$4 = 4 + 0.\mathrm{i} = (4,0)$

$1 = 1 + 0.\mathrm{i} = (1,0)$

So all real numbers are complex numbers but all complex numbers are not real numbers.

All complex numbers having imaginary parts zero are the real numbers.

## Algebra of complex numbers:

**Equality**: Two complex numbers $z = (a, b)$ *and* $w = (c, d)$ are said to be equal if and only if $a = c$ *and* $b = d$

**Addition**: The addition of two complex numbers $z = (a, b)$ *and* $w = (c, d)$ are denoted by $z + w$ and is defined as

$$z + w = (a, b) + (c, d) = (a + c, b + d)$$

**Subtraction**: The difference of two complex numbers $z = (a, b)$ *and* $w = (c, d)$ are denoted by $z - w$ and is defined as

$$z - w = (a, b) - (c, d) = (a - c, b - d)$$

**Multiplication**: The product of two complex numbers $z = (a, b)$ *and* $w = (c, d)$ are denoted by $z.w = z \times w$ and is defined as

$$z.w = (a, b).(c, d) = (ac - bd, ad + bc)$$

# Algebra of complex numbers:

## Multiplication:

$$(2,3).(4,5) = (2.4 - 3.5,\quad 2.5 + 3.4)$$

$$= (-7, 22)$$

## Division:

The division of two complex numbers z $= (a, b)$ *and* $w = (c, d) \neq 0$ are denoted by $\frac{z}{w}$ and is defined as

$$\frac{z}{w} = \frac{(a, b)}{(c,d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

Note:1 The square root of $-1$ is $i$. [ $i.e\ i^2 = -1$ or $i = \sqrt{-1}$ ]

Proof: we have

$i = (0,1)$

Now $i^2 = i \times i = (0,1) \times (0,1)$

$$= (0 \times 0 - 1 \times 1, 0 \times 1 + 1 \times 0) = (-1,0) = -1$$

$$\therefore i^2 = -1$$

Note:2

$(a,b) = a + ib$

Proof:

$$a + ib = (a,0) + (0,1)(b,0)$$

$$= (a,0) + (0 \times b - 1 \times 0, 0 \times 0 + 1 \times b)$$

$$= (a,0) + (0,b) = (a + 0, 0 + b)$$

$$= (a,b)$$

Multiplication: [Alternate]

The product of two complex numbers z $= (a, b)$ *and* $w = (c, d)$ are denoted by z. w $= z \times w$ and is defined as

$$z.w = (a, b).(c, d) = (ac - bd, ad + bc)$$

Alternate:

$$
\begin{aligned}
z.w &= (a, b).(c, d) \\
&= (a + ib).(c + id) \\
&= ac + iad + ibc + i^2 bd \\
&= ac + i(ad + bc) + (-1)bd \\
&= (ac - bd) + i(ad + bc) \\
&= (ac - bd\ , ad + bc)
\end{aligned}
$$

## Division: [Alternate]

The division of two complex numbers $z = (a, b)$ $and$ $w = (c, d) \neq 0$ are denoted by $\frac{z}{w}$ and is defined as

$$\frac{z}{w} = \frac{(a,b)}{(c,d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

Alternate:

$$\frac{z}{w} = \frac{(a,b)}{(c,d)} = \frac{a+ib}{c+id}$$

$$= \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$= \frac{ac-iad+ibc-i^2bd}{c^2-(id)^2}$$

$$= \frac{ac+ibc-iad-(-1)bd}{c^2-i^2.d^2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{(ac+bd)}{c^2+d^2} + i\,\frac{(bc-ad)}{c^2+d^2} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

# COMPLEX NUMBER

## Note:1

$(-1)^2 = 1$

$(-2)^2 = 4$

$(-1)^3 = -1$

$(-2)^3 = -8$

$$\sqrt[n]{x} = (x)^{\frac{1}{n}}$$

$$x^m \times x^n = x^{m+n}$$

$$(x^m)^n = x^{m \times n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

## Note:2

$i^2 = -1$

$i^3 = i^2.i = (-1).i = -i$

$i^4 = i^2.i^2 = (-1).(-1) = 1$

$i^5 = i^4.i = (i^2)^2.i = (-1)^2.i = 1.i = i$

$i^{11} = i^{10}.i = (i^2)^5.i = (-1)^5.i = -1.i = -i$

$i^{50} = (i^2)^{25} = (-1)^{25} = -1$

1. Find the value of following:

a) $i^{33}$

$= i^{32}.i$

$= (i^2)^{16}.i$

$= (-1)^{16}.i$

$= 1.i = i$

b. $(1+i)^2$

$= 1^2 + 2.1.i + i^2$

$= 1 + 2i + (-1)$

$= 1 + 2i - 1$

$= 2i$

c. $2\,i^2 + 6\,i^3 + 3\,i^{16} - 6\,i^{19} + 4\,i^{25}$

$= 2.(-1) + 6\,i^2.i + 3\,(i^2)^8 - 6\,i^{18}.i + 4\,i^{24}.i$

$= -2 + 6(-1).i + 3\,(-1)^8 - 6\,(i^2)^9.i + 4\,(i^2)^{12}.i$

$= -2 - 6i + 3\,.1 - 6\,(-1)^9.i + 4\,(-1)^{12}.i$

$= -2 - 6i + 3\; - 6(-1).i + 4.1.i$

$= -2 - 6i + 3 + 6i + 4i$

$= 1 + 4i$

# COMPLEX NUMBER

d. $\frac{3}{i}+\frac{2}{1+i}-\frac{4}{(1-i)^2}$

$=\frac{3}{i}+\frac{2}{1+i}-\frac{4}{1^2-2.1.i+i^2}$

$=\frac{3}{i}+\frac{2}{1+i}-\frac{4}{1-2i-1}$

$=\frac{3}{i}+\frac{2}{1+i}-\frac{4}{-2i}$

$=\frac{3}{i}+\frac{2}{1+i}+\frac{2}{i}$

$=\frac{3}{i}+\frac{2}{i}+\frac{2}{1+i}$

$=\frac{3+2}{i}+\frac{2}{1+i}$

$=\frac{5}{i}+\frac{2}{1+i}$

$=\frac{5}{i}+\frac{2}{1+i}$

$=\frac{5(1+i)+2i}{i(1+i)}$

$=\frac{5+5i+2i}{i+i^2}$

$=\frac{5+7i}{i-1}\times\frac{i+1}{i+1}$

$=\frac{5i+5+7i^2+7i}{i^2-1^2}$

$=\frac{12i+5+7(-1)}{-1-1}$

$=\frac{12i-2}{-2}$

$=\frac{12i}{-2}-\frac{2}{-2}=1-6i$

## COMPLEX NUMBER

### Example 2

2. Express each of the  of  following in the  form of A+iB:

a) $(1+3i)+(2-6i)$

$= 1+3i+2-6i$

$= 3-3i$

b) $(1+2i)(3-2i)$

$= 1\times 3-1\times 2i+2i\times 3-2i\times 2i$

$= 3-2i+6i\ -4i^2$

$= 3+4i\ -4(-1)$

$= 3+4i+4$

$= 7+4i$

# COMPLEX NUMBER

2. Express each of the of following in the form of A+iB:

c) $(1+i)^3$

$= 1^3 + 3\,.1^2.\,i + 3.\;1.\,i^2 + i^3$

$= 1 + 3i + 3.(-1) + i^2.\,i$

$= 1 + 3i - 3 + (-1).\,i$

$= -2 + 3i - i$

$= -2 - 2i$

d) $\frac{3}{2i}$

e) $\frac{5}{2-\sqrt{-1}}$

$$(a+b)^3 = a^3 + 3.a^2.b + 3.a.b^2 + b^3$$

$$(a-b)^3 = a^3 - 3.a^2.b + 3.a.b^2 - b^3$$

2. Express each of the of following in the form of A+iB:

f) $\frac{3-\sqrt{-4}}{2+\sqrt{-1}}$

$=\frac{3-\sqrt{-1\times 2^2}}{2+\sqrt{-1}}$

$=\frac{3-\sqrt{i^2\times 2^2}}{2+\sqrt{i^2}}$

$=\frac{3-2i}{2+i}\times\frac{2-i}{2-i}$

$=\frac{(3-2i)(2-i)}{2^2-i^2}$

$=\frac{3.2-3.i-2i.2+2\ .i^2}{4-(-1)}$

$=\frac{6-3i-4i.-2}{4+1}$

$=\frac{4-7i}{5}$

$=\frac{4}{5}-\frac{7i}{5}$

# COMPLEX NUMBER

2. Express each of the of following in the form of A+iB:

g) $\sqrt{\frac{1+i}{1-i}} = \sqrt{\frac{1+i}{1-i} \times \frac{1+i}{1+i}}$

$= \sqrt{\frac{(1+i)^2}{1^2-i^2}} = \sqrt{\frac{(1+i)^2}{1-(-1)}} = \sqrt{\frac{(1+i)^2}{2}} = \frac{1+i}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

h) $\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^3$

$= \left(\frac{(1-i)^2}{1^2-i^2}\right)^3 = \left(\frac{1^2-2.1.i+i^2}{1^2-i^2}\right)^3 = \left(\frac{1-2i-1}{1-(-1)}\right)^3$

$= \left(\frac{-2i}{2}\right)^3 = (-i)^3 = -i^3$

$= -i^2.i = -(-1).i = i = 0 + 1.i$

## Properties of complex numbers:

i. A real number can not be equal to an imaginary number unless each is zero.

Let and b are real numbers such that $a = ib$.

We have to show that a= 0 and b =0.

Now

$$a = ib$$

$$or, a^2 = i^2 b^2$$

$$or, a^2 = -b^2$$

$$or, a^2 + b^2 = 0$$

It is possible only when

a =0 and b =0.

Proved.

## Properties of complex numbers:

ii. If $a + ib = 0$ $then$ $a = 0$ $and$ $b = 0$.

Now

$$a + ib = 0$$

$$or, a = ib$$

$$or, a^2 = i^2 b^2$$

$$or, a^2 = -b^2$$

$$or, a^2 + b^2 = 0$$

It is possible only when

a =0 and b =0.

Proved.

## Properties of complex numbers:

iii. $a + ib = c + id \; if \; and \; only \; if \;\; then \; a = c \; and \; b = d.$

Now

$$a + ib = c + id$$

$$\Leftrightarrow \; a - c = id - ib$$

$$\Leftrightarrow \; a - c = i(d - b)$$

$$\Leftrightarrow \; (a - c)^2 = i^2(d - b)^2$$

$$\Leftrightarrow \; (a - c)^2 = -(d - b)^2$$

$$\Leftrightarrow \; (a - c)^2 + (d - b)^2 = 0$$

It is possible only when

$$a - c = 0 \; and \; d - b = 0$$

$$\Leftrightarrow a = c \; and \; d = b$$

Proved.

## Exercise

1. Evaluate:

a) $(1,0)^2$   b) $(0,1\ )^5$   c) $(0,1\ )^5$   d) $(0,1)^{11}$

Solution: (d) $(0,1)^{11}$

$$(0,1)^{11} = i^{11}$$
$$= i^{10}.\, i$$
$$= (i^2)^5.\, i$$
$$= (-1)^5.\, i = -i$$

2. Find the values of x and y in each of the following:

a) $(x, y) = (2,3) + (3,2)$   b) $(x, y) = (2,1) + (-2, -1)$

c) $(x, y) = (2,3) - (3,2)$   d) $(2,3) = (1,1) + (x, y)$

e) $(x, y) = (1,1).\,(2,3)$  f) $(x, y) = \frac{(1,1)}{(3,4)}$

2. Find the values of x and y in each of the following:

Solution: d) $(2,3) = (1,1) + (x, y)$

We have

$(2,3) = (1,1) + (x, y)$

$or,\ 2 + i3 = 1 + i + x + iy$

$or,\ 2 + i3 = (1 + x) + i(1 + y)$

Equating real and imaginary parts, we get

$2 = 1 + x\ and\ 3 = 1 + y$

$\therefore x = 1\ and\ y = 2$

2. Find the values of x and y in each of the following:

Solution: f) $(x, y) = \frac{(1,1)}{(3,4)}$

We have $(x, y) = \frac{(1,1)}{(3,4)}$

$$or,\ x + iy = \frac{1+i}{3+i4}$$

$$or,\ x + iy = \frac{1+i}{3+i4} \times \frac{3-i4}{3-i4}$$

$$or,\ x + iy = \frac{3-i4+i3-i^2 4}{3^2-(i4)^2}$$

$$or,\ x + iy = \frac{3-i+4}{9+16}$$

$$or,\ x + iy = \frac{7-i}{25} = \frac{7}{25} - i\,\frac{1}{25}$$

Equating real and imaginary parts, we get

$$\mathrm{x} = \frac{7}{25}\ and\ y = \frac{-1}{25}$$

3. Simplify :

a) $\sqrt{-9}+\sqrt{-25}-\sqrt{-36}$

b) $(3-\sqrt{-4})(2+\sqrt{-9})$

c) $\sqrt{-16}.\sqrt{-1}$

d) $3i^2+i^3+9i^4-i^7$

e) $\frac{1}{i^2}-\frac{1}{i^2}+\frac{1}{i^3}-\frac{1}{i^4}$

f) $\frac{1}{i}+\frac{1}{i^2}+\frac{1}{i^3}+\frac{1}{i^4}$

Solution: a) $\sqrt{-9}+\sqrt{-25}-\sqrt{-36}$

$$=\sqrt{i^2 9}+\sqrt{i^2 25}-\sqrt{i^2 36}$$

$$=3i+5i-6i$$

$$=2i$$

4. Prove that : $(1+i)^4.\left(1+\frac{1}{i}\right)^4 = 16$

5. Find the real numbers x and y if

a) $x + iy = (2-3i)(3-2i)$

b) $(x-1)i + (y+1) = (1+i)(4-3i)$

6. Show that $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$ is purely a real number.