

Simple harmonic motion

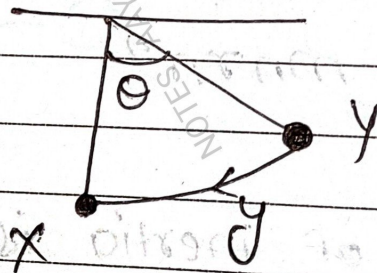
- Harmonic motion: motion in which an object repeats its path in regular interval of time.
- Restoring force: Force which brings or tries to bring towards its original position.

The restoring force is directly proportional to displacement.

Simple harmonic motion:

$$F \propto y$$
$$F = -ky$$
$$ma = -ky$$
$$a = -\frac{k}{m}y$$

$$a = -\omega^2 y$$
$$a \propto -y$$



Such type of motion in which acceleration is directly proportional to displacement and directed towards the mean position as simple harmonic motion.

i.e. $a \propto -y$

$$a = -ky$$

$$\frac{dv}{dt} = -ky$$

$$\frac{d}{dt} \left(\frac{dy}{dt} \right) = -ky$$

$$\frac{d^2y}{dt^2} = -ky$$

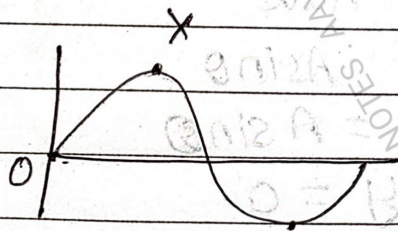
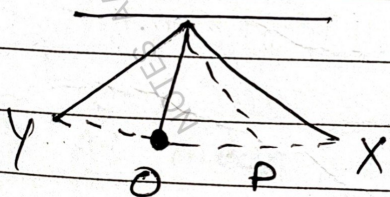
$$\frac{d^2y}{dt^2} + ky = 0$$

This is differential eqn of linear simple harmonic motion.

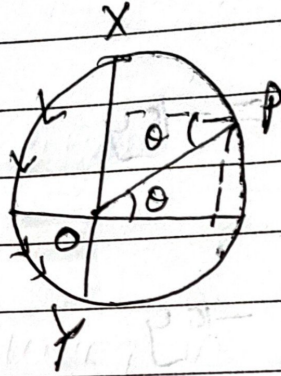
Similarly differential eqn part of angular simple harmonic motion is

$$\frac{d^2\theta}{dt^2} + k\theta = 0$$

Characteristics of simple harmonic motion:



The displacement of an object in simple harmonic motion at any instant of time is defined as the shortest distance from mean position at that time.



From ΔOMP

$$\sin \theta = \frac{OM}{OP}$$

$$\sin \theta = \frac{y}{A}$$

$$y = A \sin \theta$$

$$y = A \sin \omega t \quad \left(\because \omega = \frac{\theta}{t} \right)$$

1) At mean position we have

$$y = A \sin \theta$$

$$y = A \sin 0$$

$$y = 0$$

(ii) At extreme position

we have

$$y = A \sin \theta$$

$$y = A \sin 90^\circ$$

$$y = A \text{ (max. displacement)}$$

The maximum displacement is known as amplitude.

2) Velocity: It is defined as the rate of change of displacement.

Mathematically, velocity (v) = $\frac{dy}{dt}$

$$v = \frac{d}{dt} A \sin \omega t$$

$$v = A \frac{d}{dt} \sin \omega t$$

$$v = A(\omega \cos \omega t)$$

$$v = \omega A \cos \omega t$$

$$v = \omega A \sqrt{1 - \sin^2 \omega t}$$

$$v = \omega \sqrt{A^2 - A^2 \sin^2 \omega t}$$

$$v = \omega \sqrt{A^2 - y^2}$$

a) At mean position ($y=0$)

we have,

$$v = \omega \sqrt{A^2 - y^2}$$

$$v = \omega \sqrt{A^2 - 0} \quad (\because y=0)$$

$$v = \omega A \text{ (max velocity)}$$

b) At extreme position ($y=A$)

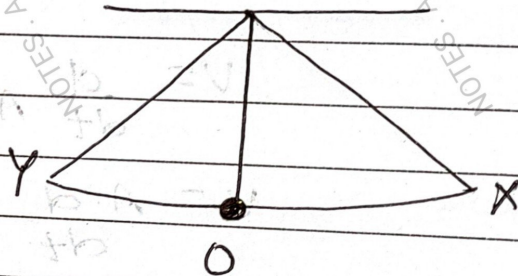
we have,

$$V = \omega \sqrt{A^2 - y^2}$$

$$V = \omega \sqrt{A^2 - A^2}$$

$$V = 0 \text{ (min. velocity)}$$

Q) An object is oscillating in simple harmonic motion with amplitude (A). Calculate its total distance and its total displacement in one complete oscillation.



$$OX + XO + OY + YO \\ \Rightarrow A + A + A + A$$

$$\text{Total distance} = 4A$$

$$\text{Displacement} = 0.$$

An object is oscillating in simple harmonic motion with amplitude (A). At what value of displacement its velocity equals to half of its maximum velocity?

$$\text{Velocity (V)} = \frac{V_{\max}}{2} = \frac{\omega A}{2}$$

$$\text{Displacement (y)} = ?$$

We have

$$V = \omega \sqrt{A^2 - y^2}$$

$$\frac{\omega A}{2} = \omega \sqrt{A^2 - y^2}$$

squaring,

$$\frac{A^2}{4} = A^2 - y^2$$

$$y^2 = A^2 - \frac{A^2}{4} = \frac{3A^2}{4}$$

$$y = \frac{\sqrt{3} A}{2}$$

Acceleration: It is defined as the rate of change of velocity.

Mathematically,

$$\text{acceleration (a)} = \frac{dv}{dt}$$

$$a = \frac{d}{dt} A \omega \cos \omega t$$

$$a = A \omega \frac{d}{dt} \cos \omega t$$

$$a = A \omega (-\omega \sin \omega t)$$

$$a = -\omega^2 A \sin \omega t$$

$$\boxed{a = -\omega^2 y} \quad (\because y = A \sin \omega t)$$

(a) At mean position ($y=0$)

we have

$$a = -\omega^2 y$$

$$a = 0 (\because y=0)$$

(b) At extreme position ($y=A$)

we have,

$$a = -\omega^2 y$$

$$a = -\omega^2 A (\because y=A) \text{ (max acc.)}$$

Time period: Defined as the time taken to make one complete oscillation. It is denoted by 'T'.

$$\text{The Time period (T)} = \frac{2\pi}{\omega}$$

we have

$$\text{Acc(A)} = \omega^2 y$$

$$\omega^2 = \frac{a}{y}$$

$$\omega = \sqrt{\frac{a}{y}}$$

$$\text{Time period (T)} = \frac{2\pi}{\sqrt{a/y}} = 2\pi \sqrt{y/a}$$

$$T = 2\pi \sqrt{\frac{\text{Displacement}}{\text{acceleration}}}$$

Frequency: It is defined as the number of complete oscillation net per second. It is denoted by 'f' or 'n'.

Mathematically it is defined as the reciprocal of time period.

$$\text{Frequency } (f) = \frac{1}{T}$$

$$f = \frac{1}{2\pi \sqrt{\frac{y}{g}}}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{y}}$$

Its unit is s^{-1} or H_z

Q) The eqn of displacement of an object in simple harmonic motion is

$$y = 0.2 \sin 100\pi t$$

The displacement is in cm and time is in second.

calculate (i) Amplitude

(ii) Angular velocity

(iii) Time period

(iv) frequency

(v) maximum velocity and maximum acc.

Soln,

here

$$y = 0.2 \sin 100\pi t \quad \text{--- (i)}$$

equation of displacement is

$$y = A \sin \omega t \quad \text{--- (ii)}$$

comparing eqn (i) and (ii)

$$\text{(i)} \quad A = 0.2 \text{ cm}$$

$$\text{(ii)} \quad \omega = 100\pi$$

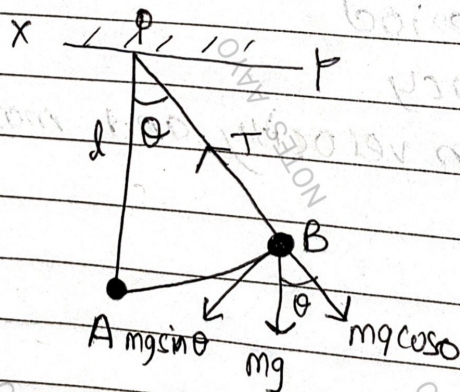
$$\text{(iii)} \quad T = \frac{2\pi}{\omega} = \frac{1}{50} \text{ s}$$

$$\text{(iv)} \quad f = \frac{1}{T} = 50 \text{ Hz}$$

$$\text{(v)} \quad V_{\max} = \omega A \quad \text{and} \quad a_{\max} = \omega^2 A$$

Simple pendulum

A heavy point mass suspended from a rigid support with the help of weightless inextensible inextensible and perfectly flexible string which can oscillate freely is known as simple pendulum.



let a simple pendulum of mass 'm' is suspended from point P of rigid support x y

let $PA = l$ be the effective length of simple pendulum.

now, the pendulum is displaced from its initial position A to B through small linear displacement y . And angular displacement θ . When the pendulum is at point B, its weight mg is acting vertically downward and tension (T) is acting towards the fixed point P. On resolving weight mg into its constituent components we get $mg \cos \theta$ and $mg \sin \theta$ as shown in figure.

Component $mg \cos \theta$ is balanced by tension (T)

$$\text{i.e. } T = mg \cos \theta$$

The component $mg \sin \theta$ acting towards the mean position provides restoring force.

$$\text{Restoring force (F)} = -mg \sin \theta$$

$$ma = -mg \sin \theta$$

$$a = -g \sin \theta$$

For small angle $\sin \theta \approx \theta$

$$a \approx -g \sin \theta$$

From fig $\theta = \frac{\text{Arc AB}}{PA}$

For small angle $\text{Arc } AB \approx \text{cord } AB$

$$\theta = \frac{AB}{PA}$$

$$\theta = y/l \quad \left(\begin{array}{l} AB=y \\ PA=l \end{array} \right)$$

now eqn (1) becomes

$$a = -g \frac{y}{l}$$

$$a = -ky \quad (\because k = \frac{g}{l} \text{ is constant})$$

$$\boxed{a \propto -y}$$

Above result shows that motion of simple pendulum is simple harmonic.

Time period of simple pendulum is

$$\text{Time period } (T) = 2\pi \sqrt{\frac{\text{Disp}}{\text{Acc.}}}$$

$$T = 2\pi \sqrt{\frac{y}{g y/l}}$$

$$T = 2\pi \sqrt{\frac{l}{g}} \Rightarrow T \propto \sqrt{l}$$

$$T \propto \sqrt{\frac{l}{g}}$$

The time period of simple pendulum is independent to its mass and amplitude of oscillation.

In gravity free space time period of simple pendulum is infinite. That means it does not oscillate.

Second's pendulum $\rightarrow$ A simple pendulum having time period 2 seconds is known as second's pendulum.

Time period of a simple pendulum is $T = 2\pi \sqrt{\frac{l}{g}}$

For second's pendulum, $T = 2 \text{ sec.}$

$$T = 2\pi \sqrt{\frac{l}{g}}$$

Sq. both sides,

$$1 = \frac{\pi^2 l}{g}$$

$$l = \frac{g}{\pi^2}$$

$$= \frac{9.8}{(3.14)^2}$$

$$= 0.993 \text{ m.} = 99.3 \text{ cm.}$$

A simple pendulum of length 99.3 cm is also known as second's pendulum.

Q) A simple pendulum is oscillating with time period T . Calculate % change in time period when its length is increased by 4% b) its mass is increased by 30%.

Soln:

original length $l_1 = l$

Time period $T_1 = T$

now,

1st case:

$$T_1 = 2\pi \sqrt{\frac{l_1}{g}}$$

$$T = 2\pi \sqrt{\frac{l}{g}} \quad \text{--- (1)}$$

2nd case:

$$T_1 = 2\pi \sqrt{\frac{l_2}{g}} = 2\pi \sqrt{\frac{1.04l}{g}}$$

$$T_1 = 2\pi \sqrt{\frac{l}{g}} \sqrt{1.04}$$

$$T_1 = T \times 1.02$$

$$\text{now \% change} = \frac{T_1 - T}{T} \times 100\%$$

$$= \frac{1.02T - T}{T} \times 100\%$$

$$= 2\%$$

b) Time period of simple pendulum is independent to mass.

{ When length of simple pendulum is increased by $N\%$, its time period increases by $\sqrt{N}\%$.

Q) Time period of a simple pendulum at surface of Earth is 4s. Calculate its time period at surface of moon.

⇒ For Earth's surface, $T = 2\pi \sqrt{\frac{l}{g}}$

$$4 = 2\pi \sqrt{\frac{l}{g}} \quad \text{--- (i)}$$

For moon,

$$T_m = 2\pi \sqrt{\frac{l}{g}} = 2\pi \sqrt{\frac{l}{g/6}}$$

$$T_m = 2\pi \sqrt{\frac{6l}{g}} \quad \text{--- (ii)}$$

From (i) and (ii)

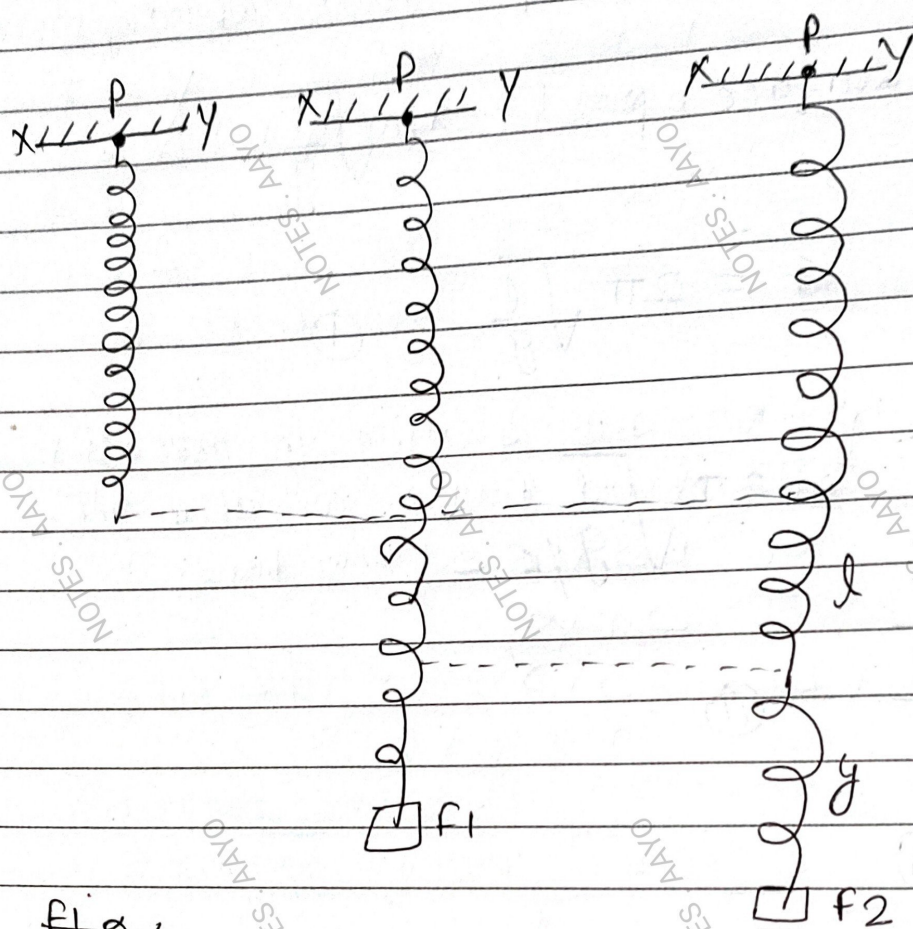
$$\frac{T_m}{4} = \frac{2\pi \sqrt{\frac{6l}{g}}}{2\pi \sqrt{\frac{l}{g}}}$$

$$T_m = 4\sqrt{6} \text{ sec}$$

Q) Time period of a simple pendulum is $\sqrt{6}$ times greater than surface of earth

Q) An object is oscillating in simple harmonic motion with time period T and amplitude A . Calculate time period taken to complete half of its amplitude starting from mean position.

Motion of Helical spring (vertical spring)



Fig

f2

Let a spring of length L is suspended from point P of rigid support XY . The spring is elongated by applying force f_1 . Due to which elongation produced is l . Since we know that the elongation produced is directly proportional to length force.

i.e. $f_1 \propto l$

$$f_1 = -kl \quad \text{--- (1)}$$

$f_2 \propto (l+y)$ now the spring is further elongated by y on applying force f_2 . Now the total elongation becomes, $l+y$

f_2 since we know that $f_2 \propto l+y$

$$f_2 = -k(l+y) \quad \text{--- (1)}$$

The force responsible to produce elongation by y only is $f = f_2 - f_1$

$$F = -kx - ky + kx$$

$$ma = -ky$$

$$a = \frac{-k}{m} \cdot y$$

$$a = -k'y \quad (\because k' = \frac{k}{m} \text{ is constant})$$

$$\boxed{a \propto -y}$$

The above result shows that acceleration is directly proportional to displacement and directed towards mean position. So motion of helical spring is simple harmonic.

now, its time period is

$$\text{Time period (T)} = 2\pi \sqrt{\frac{\text{Disp}}{a_{\text{cen}}}} = 2\pi \sqrt{\frac{y}{a}}$$

$$T = 2\pi \sqrt{\frac{y}{y \frac{k}{m}}}$$

$$\boxed{T = 2\pi \sqrt{\frac{m}{k}}}$$

Factors effecting spring constant or Force constant:

From the expression of Young modulus of elasticity,

$$F = \frac{YAx}{l} \quad \text{--- (i)} \quad \left(\begin{array}{l} x = \text{elongation} \\ l = \text{original length} \end{array} \right)$$

We know,

$$F = kx \quad \text{--- (ii)}$$

From (i) and (ii)

$$k = \frac{YA}{l}$$

- (i) $k \propto Y$ (if A and l are constant)
- (ii) $k \propto A$ (if Y and l are constant)
- (iii) $k \propto \frac{1}{l}$ (Y and A are constant)

Q) A wire of length l and spring constant k is cut into 'n' equal parts. The spring constant of each part will be

a) k

b) nk

c) $\frac{k}{n}$

d) $n^2 k$

$$l_1 = l$$

$$l_2 = \frac{l}{n}$$

$$k_1 = k$$

$$k_2 = ?$$

we have,

$$k \propto \frac{1}{l}$$

$$\frac{k_2}{k_1} = \frac{l_1}{l_2} = \frac{k}{1/n}$$

$$k_2 = nk_1 = nk$$

Q) An object of mass 2 kg is suspended at one end of a spring. Due to which elongation produced is 0.2 m . Calculate Force Constant and time period.