

Rotational Dynamics

Rigid body: Such type of body whose intermolecular separation always remains constant when it is in translational or rotational motion.

Moment of inertia: It is defined as the algebraic sum of product of mass and square of perpendicular distance from axis of rotation.

$$I = \sum_{i=1}^n m_i r_i^2$$

It is denoted by ~~capital~~ I . Mathematically,

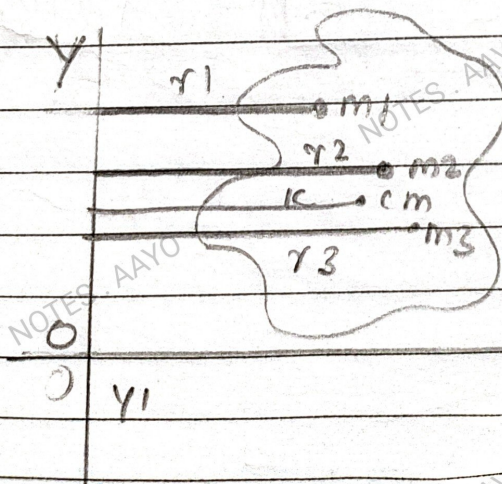
$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2$$

Its unit is kgm^2

Moment of Inertia depends upon mass, its distance from axis of rotation and distribution of mass.

Radius of gyration: It is defined as the perpendicular distance between center of mass and radius of gyration axis of rotation.

It is denoted by k .



Let a rigid body consisting of large number of elementary particles having masses $m_1, m_2, m_3, \dots$ with their respective perpendicular distances $r_1, r_2, r_3, \dots, r_n$. Let M be the total mass of rigid body and radius of gyration is k . Now moment of inertia of the rigid body is.

$$\text{moment of inertia (I)} = Mk^2 \quad \text{--- (1)}$$

From the definition of moment of inertia,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2$$

$$\text{Let } m_1 = m_2 = m_3 = \dots = m_n (= m \text{ say})$$

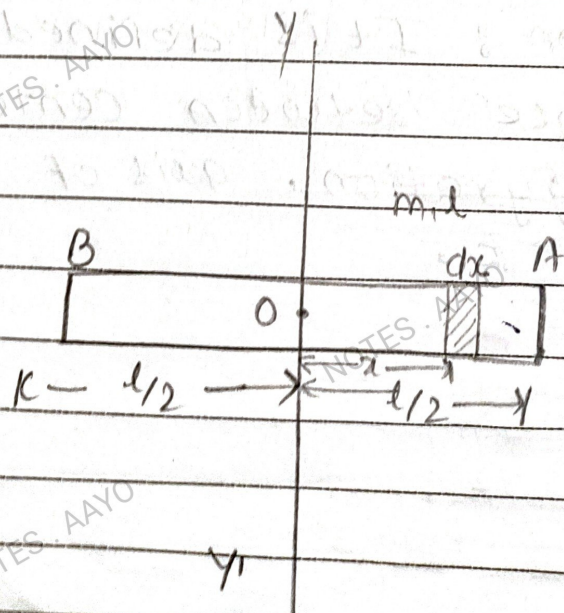
$$I = m r_1^2 + m r_2^2 + m r_3^2 + \dots + m r_n^2$$

$$I n k^2 = m (r_1^2 + r_2^2 + r_3^2 + \dots + r_n^2)$$

$$n k^2 = m (r_1^2 + r_2^2 + r_3^2 + \dots + r_n^2)$$

$$k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \dots + r_n^2}{n}}$$

Moment of inertia of a uniform rod axis of rotation passing through center and $\perp$ to length



let us take uniform rod AB of
~~about~~ is the rod is rotating ~~about~~ along its
axis yoy'

let us take a small element of length dx at
a distance x from center O.

$$\text{mass per unit length} = \frac{m}{l}$$

$$\text{mass of small element } (dm) = \frac{m}{l} dx$$

M.I (moment of inertia) of small element

$$d(I) = dm x^2$$

$$dI = \frac{m}{l} dx x^2$$

$$dI = \frac{m}{l} x^2 dx \quad \text{--- (1)}$$

now, moment of inertia of rod AB can be obtained
by integrating equation 1 from limit $-\frac{l}{2}$ to $\frac{l}{2}$

$$I = \int_{-l/2}^{l/2} dI = \frac{m}{l} \int_{-l/2}^{l/2} x^2 dx$$

$$I = \frac{m}{l} \left| \frac{x^3}{3} \right|_{-l/2}^{l/2}$$

$$I = \frac{m}{3l} \left[x^3 \right]_{-l/2}^{l/2}$$

$$I = \frac{m}{3l} \left[\left(\frac{l}{2} \right)^3 - \left(-\frac{l}{2} \right)^3 \right]$$

$$I = \frac{m}{3l} \left(\frac{l^3}{8} + \frac{l^3}{8} \right)$$

$$= \frac{m}{3l} \cdot \frac{2l^3}{8}$$

$$I = \frac{1}{12} ml^2$$

note: moment of inertia of uniform rod axis of rotation passing through one end and $\perp$ to length is $I = \frac{1}{3} ml^2$
moment of inertia is a tensor quantity.

Torque

It is defined as the turning effect of force. mathematically,

$$T = \text{Force} \times \text{perp. distance from axis of rotation}$$

It is denoted by τ

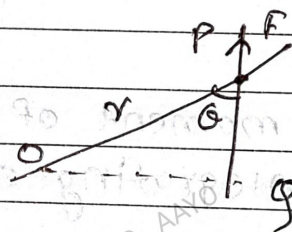
$$\tau = F \times OG$$

$$\tau = F (OP \sin \theta)$$

From ΔOGP

$$\tau = Fr \sin \theta$$

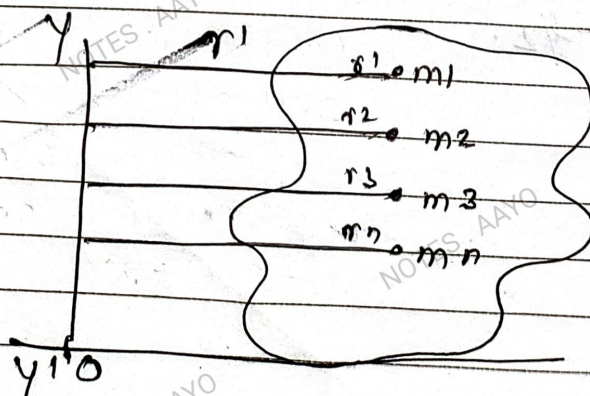
$$\sin \theta = \frac{OG}{OP}$$



In vector form $[\vec{\tau} = \vec{r} \times \vec{F}]$ $OG = OP \sin \theta$
vector angle $= 90^\circ$

The value of Torque is maximum if angle is 90° and minimum if angle is 0° .

Relation between Torque, moment of Inertia, and angular acceleration,



let the rigid bodies is rotating with constant angular acceleration

now, force on each particle is

$$F_1 = m_1 a_1, F_2 = m_2 a_2, F_3 = m_3 a_3 \dots F_n = m_n a_n$$
$$F_1 = m_1 \alpha r_1, F_2 = m_2 \alpha r_2, F_3 = m_3 \alpha r_3 \dots F_n = m_n \alpha r_n$$

now, torque on each particle is,

$$T_1 = F_1 r_1, T_2 = F_2 r_2, T_3 = F_3 r_3, \dots T_n = F_n r_n$$
$$T_1 = m_1 \alpha r_1^2, T_2 = m_2 \alpha r_2^2, T_3 = m_3 \alpha r_3^2, \dots T_n = m_n \alpha r_n^2$$
$$\text{Total torque} = T_1 + T_2 + T_3 + \dots T_n$$
$$T = m_1 \alpha r_1^2 + m_2 \alpha r_2^2 + m_3 \alpha r_3^2 + \dots m_n \alpha r_n^2$$
$$T = (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots m_n r_n^2) \alpha$$

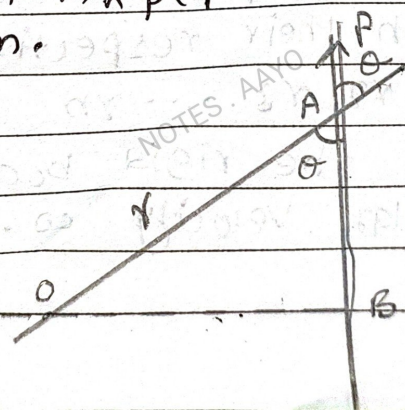
$$\boxed{T = I \alpha} \because I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots m_n r_n^2$$

The above result shows that torque is product of moment of inertia and angular acceleration.

Angular momentum: It is defined as the product of linear momentum and perpendicular distance from axis of rotation. It is denoted by L .

It is also known as moment of linear momentum. mathematically,

angular momentum (L) = linear momentum $\times$ perp. distance from axis of rotation.



$$l = p \times (OB)$$

$$l = p \times (OA \sin \theta)$$

$$l = pr \sin \theta \quad (\because OA = r)$$

$$l = mvr \sin \theta \quad (p = mv)$$

From $\triangle OBA$

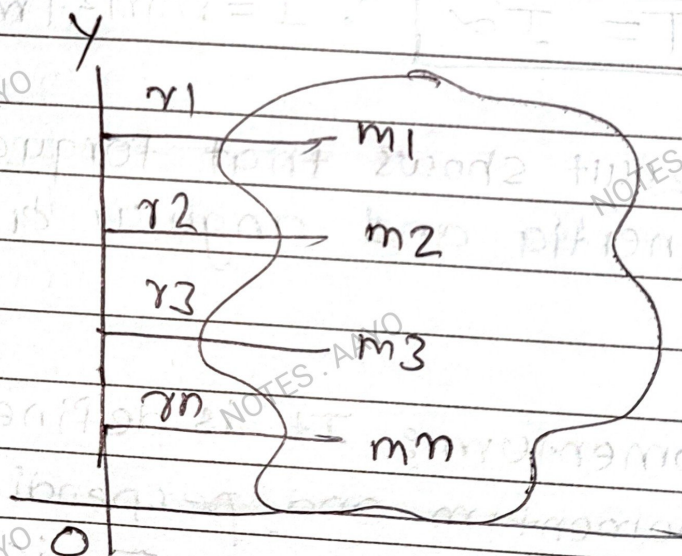
$$\sin \theta = \frac{OB}{OA}$$

In vector Form

$$\vec{l} = \vec{r} \times \vec{p}$$

The value of angular momentum is maximum if angle is 90° and minimum if angle is 0° .

Relation between angular momentum, moment of ~~in~~ertia, and angular velocity



let a rigid body consisting of large number element particles having masses $m_1, m_2, m_3, \dots, m_n$ with their respective perpendicular distance $r_1, r_2, r_3, \dots, r_n$

let the rigid body is rotating with constant angular velocity ω .

now, linear velocity of each elementary particle is

$$v_1 = \omega r_1, v_2 = \omega r_2, v_3 = \omega r_3 \dots v_n = \omega r_n$$

linear momentum of each particle is

$$p_1 = m_1 v_1, p_2 = m_2 v_2, p_3 = m_3 v_3 \dots p_n = m_n v_n$$

$$p_1 = m_1 \omega r_1, p_2 = m_2 \omega r_2, p_3 = m_3 \omega r_3 \dots p_n = m_n \omega r_n$$

Angular mom. of each particle is,

$$J_1 = p_1 r_1, J_2 = p_2 r_2, J_3 = p_3 r_3 \dots J_n = p_n r_n$$

$$J_1 = m_1 \omega r_1^2, J_2 = m_2 \omega r_2^2, J_3 = m_3 \omega r_3^2 \dots J_n = m_n \omega r_n^2$$

now, total ~~momentum~~ angular mom is,

$$J = J_1 + J_2 + J_3 \dots + J_n$$

$$J = m_1 \omega r_1^2 + m_2 \omega r_2^2 + m_3 \omega r_3^2 + \dots + m_n \omega r_n^2$$

$$J = (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2) \omega$$

$$J = I \omega \quad (I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2)$$

The above result shows that angular momentum is product of linear momentum inertia and angular velocity

Relation between angular momentum and torque

Angular momentum of a rigid body of moment of inertia (I) rotating with angular velocity ω is

$$\text{Angular momentum } (J) = I \omega$$

Diff. both sides w.r.t. to time.

$$\frac{dJ}{dt} = \frac{d}{dt} (I \omega)$$

$$\text{or, } \frac{dl}{dt} = I \frac{d\omega}{dt}$$

$$\text{or, } \frac{dl}{dt} = I \alpha \left(\because \alpha = \frac{d\omega}{dt} \right)$$

$$\frac{dl}{dt} = \tau$$

$$\boxed{\tau = \frac{dl}{dt}}$$

now, Torque can also be defined as the rate of change of angular momentum.

Conservation of angular momentum

It states the in absence of external torque, total angular momentum of an insulated system always remains constant.

proof:

Since we know that rate of change of angular momentum is equal to torque.

$$\text{i.e. Torque } (\tau) = \frac{dl}{dt}$$

If there is no external torque,

$$\text{i.e. } \tau = 0, \quad \frac{dl}{dt} = 0$$

on integrating both sides,

$$\int \frac{d}{dt} l = \int 0$$

$l = \text{constant.}$

$I\omega = \text{constant}$

$$I_1 \omega_1 = I_2 \omega_2$$

$$* I\omega = k \quad * I = mr^2$$

$$\omega \propto \frac{1}{I}$$

$$* I\omega = k$$

$$I \cdot 2\pi f = k$$

$$f \propto \frac{1}{I}$$

$$* I\omega \propto k$$

$$I \cdot \frac{2\pi}{T} = k$$

$$I \propto T$$

$$T \propto I$$

Q) A disc of moment of inertia $5 \times 10^{-4} \text{ kg m}^2$ is rotating with 40 rpm (rev. per minute). If some wax of mass 0.02 kg is placed at a distance 0.08 m from center. Calculate new rate of revolution per minute

Soln: moment of inertia of disc (I) = $5 \times 10^{-4} \text{ kg m}^2$

$$\text{no. of revolution} = 40 \text{ rpm} = \frac{40}{60} \text{ rps} = \frac{2}{3} \text{ rps}$$

$$\omega = 2\pi f = \frac{4\pi}{3} \text{ rad s}^{-1}$$

When wax is dropped,

$$\text{mass of wax (m)} = 0.02 \text{ kg}$$

$$\text{moment of Inertia of wax (I)} = mr^2 = 0.02 \times (0.08)^2 = 1.28 \times 10^{-4} \text{ kg m}^2$$

let f_1 be the new rate of revolution and ω_1 be new angular velocity.

$$\begin{aligned}\text{initial angular momentum of disc} &= I_0 \omega_1 + I \omega_1 \\ &= (I_0 + I) \omega_1 = (I_0 + I) \times 2\pi f_1\end{aligned}$$

Then by principle of conservation of angular momentum,

$$I_0 \omega = (I_0 + I) 2\pi f_1$$

$$5 \times 10^{-4} \times \frac{4\pi}{3} = (5 \times 10^{-4} + 1.28 \times 10^{-4}) 2\pi f_1$$

$$\begin{aligned}f_1 &= 0.53 \text{ rev. per second.} \\ &= 32 \text{ rev. per minute.}\end{aligned}$$

Spr:

$$I_1 = 5 \times 10^{-4} \text{ kgm}^2$$

$$f_1 = 40 \text{ rpm}$$

$$m = 0.02 \text{ kg} \quad r = 0.08 \text{ m} \quad f_2 = ?$$

$$\text{final } mI (I_2) = (I_1 + m r^2) =$$

from conservation of angular mom,

$$I_1 \omega_1 = I_2 \omega_2$$

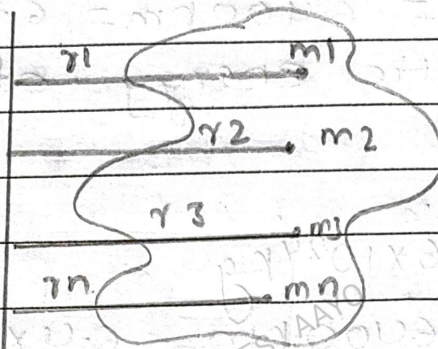
$$I_1 2\pi f_1 = I_2 2\pi f_2$$

$$f_2 = 32 \text{ rpm}$$

kinetic energy of rotating body.

let a rigid body consisting of large number of elementary particles having masses $m_1, m_2, m_3, \dots, m_n$ with their respective perpendicular distance from axis of rotation $r_1, r_2, r_3, \dots, r_n$.

let the rigid body is rotating with constant angular velocity (ω).
The linear velocity of each particle is



$$v_1 = \omega r_1, v_2 = \omega r_2, v_3 = \omega r_3, \dots, v_n = \omega r_n$$

K.E of each particle is

$$E_1 = \frac{1}{2} m_1 v_1^2, E_2 = \frac{1}{2} m_2 v_2^2, E_3 = \frac{1}{2} m_3 v_3^2, \dots, E_n = \frac{1}{2} m_n v_n^2$$

$$E_1 = \frac{1}{2} m_1 \omega^2 r_1^2, E_2 = \frac{1}{2} m_2 \omega^2 r_2^2, E_3 = \frac{1}{2} m_3 \omega^2 r_3^2, \dots, E_n = \frac{1}{2} m_n \omega^2 r_n^2$$

Total K.E is,

$$E = E_1 + E_2 + E_3 + \dots + E_n$$

$$E = \frac{1}{2} m_1 \omega^2 r_1^2 + \frac{1}{2} m_2 \omega^2 r_2^2 + \frac{1}{2} m_3 \omega^2 r_3^2 + \dots + \frac{1}{2} m_n \omega^2 r_n^2$$

$$E = \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2) \omega^2$$

$$E = \frac{1}{2} I \omega^2 \quad \left(\because I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2 \right)$$

m.I of solid sphere

$$I = \frac{2}{5} m R^2$$

Date _____
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Q) Assuming earth to be a complete sphere of radius 6400 km and mass 6×10^{24} kg

$R = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$. Calculate rotational kinetic energy of earth.

Given:

$$m = 6 \times 10^{24} \text{ kg}$$

$$R = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$$

We have,

m.I of earth

$$I = \frac{2}{5} m R^2 = \frac{2}{5} \times 6 \times 10^{24} \times (6.4 \times 10^6)^2$$
$$= 9.8304 \times 10^{37}$$

now,

$$\omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{86400} = \frac{2 \times 3.14}{86400}$$

$$K.E = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} \times \frac{2}{5} \times \left(\frac{2 \times 3.14}{86400} \right)^2$$

$$= \frac{5 \times 2}{10} \times 1 = 9.8304 \times 10^{37}$$

kinetic energy of rolling body.

let a rigid body of mass 'm' and moment of inertia 'I'.
let its radius is (R) and radius of gyration is (K)

let the rigid body is moving with angular velocity (ω) and linear velocity (v). Now, kinetic energy of rolling body is given by sum of translational kinetic energy and rotational kinetic energy.

$$(K.E)_{\text{rolling}} = (K.E)_{\text{translational}} + (K.E)_{\text{rot.}}$$

$$(K.E)_{\text{rolling}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$(K.E)_{\text{rolling}} = \frac{1}{2}mv^2 + \frac{1}{2}mK^2 \cdot \frac{v^2}{R^2} \left(\because \omega = \frac{v}{R} \text{ } I = mR^2 \right)$$

$$(K.E)_{\text{rolling}} = \frac{1}{2}mv^2 \left(1 + \frac{K^2}{R^2} \right)$$

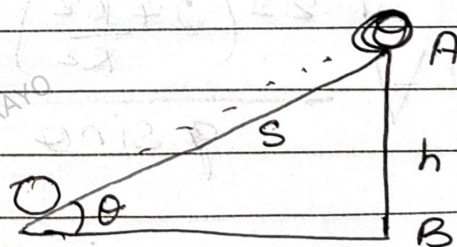
Acceleration of an object rolling down along a smooth incline

From conservation of energy,

Gain in rolling K.E = loss in P.E

$$\frac{1}{2}mv^2 \left(1 + \frac{K^2}{R^2} \right) = mgh$$

$$v^2 = \frac{2gh}{1 + \frac{K^2}{R^2}}$$



$$V = \sqrt{\frac{2gs \sin \theta}{\left(1 + \frac{k^2}{R^2}\right)}} \quad \left(\begin{array}{l} \text{From } \triangle ABC \\ \sin \theta = h/s \end{array} \right)$$

we have,

$$V^2 = u^2 + 2as$$

$$\frac{2gs \sin \theta}{\left(1 + \frac{k^2}{R^2}\right)} = 2as$$

$$a = \frac{g \sin \theta}{1 + \frac{k^2}{R^2}}$$

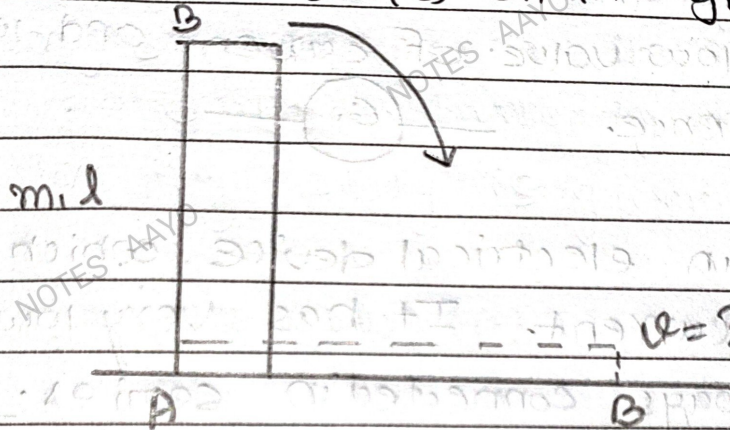
now, For time to reach C from A,

$$s = ut + \frac{1}{2} at^2$$

$$s = \frac{1}{2} \cdot \frac{g \sin \theta}{1 + \frac{k^2}{R^2}} \cdot t^2$$

$$t = \sqrt{\frac{2s \left(1 + \frac{k^2}{R^2}\right)}{g \sin \theta}}$$

A uniform rod of mass m and length l , standing vertically falls down without slipping. Calculate its linear velocity when it strikes on the ground.



From conservation of energy,
gain in Kinetic energy = Loss in P.E

$$\frac{1}{2} I \omega^2 = mgl$$

$$\frac{1}{2} I \omega^2 = mgl$$

$$\frac{1}{2} \times \frac{1}{3} m l^2 \times \frac{v^2}{l^2} = \frac{mgl}{2}$$

$$v = \sqrt{3gl}$$